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An Atypical Example of Stability and Instability*

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1. A THEOREM ON STABILITY

We consider the linear system of ordinary differential equations,

$$\frac{dz}{dt} = Az, \qquad z \in \mathbb{R}^n, \qquad t \geqslant 0,$$

the coefficient matrix $A = (A_{ij}(t))$ being defined by $A_{ij}(t) = -a_i(t)\, b_j(t)$ where $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ are vector valued functions for $t \geqslant 0$ (cf. [1] in Notes and References). The system can be rewritten as follows:

$$\frac{dz}{dt} = -a(b \cdot z). \tag{1}$$

Here $b \cdot z = \sum_{i=1}^n b_i z_i$ and we shall denote by $|z|$ the usual Euclidean norm, i.e.

$$|z| = \sqrt{z \cdot z}.$$

We assume throughout that

$$a_i(t) \geqslant 0 \qquad \text{and} \qquad b_i(t) \geqslant 0, \qquad i = 1, ..., n,$$

and seek additional conditions on $a(t)$ and $b(t)$ which insure that the linear system (1) is stable on the right or, equivalently, is bounded:

$$|z(t)| \leqslant \text{const}\, |z(0)|. \tag{2}$$

For these notions the reader can consult reference [2].

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One checks easily that zero is an eigenvalue of the coefficient matrix A with multiplicity $(n-1)$ and that the only possibly nonzero eigenvalue is the trace of $A(t) = -a(t) \cdot b(t)$. Therefore, the simplest stability results, concerning negative real parts of eigenvalues, are not applicable.

It is in fact easy to given examples exhibiting instability. For example, $z_1(t) = t - 1 + e^{-t}$, $z_2(t) = -1$ is an unbounded solution of (1) in the case $n = 2$ corresponding to the choice $a = (1, 0)$ and $b = (1, t)$. The instability in this case is due to the orthogonality, in the limit, of the vectors a and b, i.e.

$$\lim_{t\to+\infty} \frac{a \cdot b}{|a|\,|b|} = 0, \tag{3}$$

and is not otherwise related to the fact that the coefficient matrix is unbounded. In fact, the choice $a = (1, 0)$, $b = (0, 1)$ again leads to instability for the same reason while the choice $a = (\alpha(t), \alpha(t))$ and b arbitrarily chosen leads to stability whether or not $|a(t)|$ and $|b(t)|$ are bounded. In this latter example we have, in contrast to (3), the following estimate,

$$\frac{a(t) \cdot b(t)}{|a(t)|\,|b(t)|} \geqslant K^{-1}, \qquad t \geqslant 0 \tag{4}$$

where $K = \sqrt{n}$ in the analogous n-dimensional example. However, even when (4) is satisfied instability may still result due to variations in the directions of both vectors $a(t)$ and $b(t)$. In Section 2 we give an example of such instability.

We conjecture that condition (4) together with the bounded variation over $0 \leqslant t < \infty$ of either of the vectors $a(t)\,|a(t)|^{-1}$ or $b(t)\,|b(t)|^{-1}$ implies the stability of (1). (By the variation of the vector $c(t)$ over $[t_1, t_2]$ we mean

$$\int_{t_1}^{t_2} \left| \frac{dc}{dt} \right| dt.)$$

We prove somewhat less than the conjecture, namely that (1) is stable if (4) is satisfied and if *all* the ratios $a_i(t)/a_j(t)$ are of bounded variation over $0 \leqslant t < \infty$.

THEOREM. *Let* (i) $a_i(t)$, $i = 1,\ldots, n$ *be absolutely continuous positive functions defined on* $t \geqslant 0$;

(ii)
$$\int_0^\infty \left| \frac{d}{dt} \frac{a_i}{a_j} \right| dt < \infty \tag{5}$$

for all $i, j = 1, 2,\ldots, n$; (iii) $b_i(t)$, $i = 1,\ldots, n$, *be locally summable nonnegative* *functions on* $t \geqslant 0$. *Then the linear system* (1) *is stable on the right* i.e. *the estimate* (2) *is valid.*

REMARK. Condition (5) implies both (4) and the bounded variation of $a(t) \mid a(t) \mid^{-1}$. In fact, because of (5) we have for all i and j and for all $t \geqslant 0$ the estimate

$$a_i(t) \leqslant L a_j(t) \tag{6}$$

where

$$L = \max_{i,j} \left(\left| \frac{a_i(0)}{a_j(0)} \right| + \int_0^\infty \left| \frac{d}{dt} \frac{a_i}{a_j} \right| dt \right).$$

Therefore,

$$\frac{a \cdot b}{\mid a \mid \mid b \mid} \geqslant \frac{a_i}{L \mid a \mid} \frac{\Sigma b_j}{\mid b \mid} \geqslant \frac{a_i}{L \mid a \mid} \geqslant \frac{1}{L^2} \sqrt{n}$$

so that (4) is satisfied with $K = L^2 \sqrt{n}$. Moreover,

$$\left| \frac{d}{dt} \frac{a_i}{\mid a \mid} \right| = \left| \sum_j \frac{a_j^3}{\mid a \mid^3} \frac{d}{dt} \left(\frac{a_i}{a_j} \right) \right| \leqslant \sum_j \frac{a_j^3}{\mid a \mid^3} \left| \frac{d}{dt} \left(\frac{a_i}{a_j} \right) \right|$$

$$\leqslant \sum_j \left| \frac{d}{dt} \frac{a_i}{a_j} \right|$$

so that

$$\int_0^\infty \left| \frac{d}{dt} \frac{a}{\mid a \mid} \right| dt < \infty. \tag{7}$$

Therefore, condition (5) implies (6) and (7). Although we shall not make use of the fact we should point out that (5) is actually *equivalent* to (6) and (7). This is easily verified using the following identity:

$$\frac{d}{dt} \left(\frac{a_i}{a_j} \right) = \frac{\mid a \mid}{a_j} \frac{d}{dt} \left(\frac{a_i}{\mid a \mid} \right) - \frac{a_i \mid a \mid}{a_j a_j} \frac{d}{dt} \left(\frac{a_j}{\mid a \mid} \right).$$

PROOF OF THE THEOREM. If $z(t)$ is any solution of (1) then it follows from the strict positivity of each component of $a(t)$ that for any i, j

$$\frac{dz_j}{dt} = \frac{a_j}{a_i} \frac{dz_i}{dt}.$$

Upon integrating this equation and using integration by parts we obtain

$$z_j(t) = \frac{a_j(t)\, z_i(t)}{a_i(t)} - \gamma_{ij} - \int_0^t z_i(s) \frac{d}{ds} \left(\frac{a_j}{a_i} \right) ds \tag{8}$$

where $\gamma_{ij} = [z_i(0)\, a_j(0) - z_j(0)\, a_i(0)]\, (a_i(0))^{-1}$. Fixing the value of i and letting j take on all values $1,..., n$, equation (8) then gives a representation of $z(t)$ in terms of the single component z_i. We substitute this in the right side of (1) and then write out the ith component of equation (1) to obtain

$$\frac{dz_i}{dt} + (a \cdot b)\, z_i = a_i \sum_{j \neq i} b_j \left[\gamma_{ij} + \int_0^t z_i(s) \frac{d}{ds}\left(\frac{a_j(s)}{a_i(s)}\right) ds\right],$$

which can be integrated to give

$$z_i(t) \exp\left\{\int_0^t a \cdot b\, ds\right\} = z_i(0) + C_i + D_i \tag{9}$$

where

$$C_i = \int_0^t \exp\left\{\int_0^s a \cdot b\right\} a_i(s) \sum_{j \neq i} b_j(s)\, \gamma_{ij}\, ds$$

and

$$D_i = \int_0^t \exp\left\{\int_0^s a \cdot b\right\} a_i(s) \sum_{j \neq i} b_j(s) \int_0^s z_i(\tau) \frac{d}{d\tau}\left(\frac{a_j(\tau)}{a_i(\tau)}\right) d\tau\, ds.$$

Now (6) implies that $a_i \sum_j b_j \leqslant L(a \cdot b)$ so that if $\gamma_i = \max_j |\gamma_{ij}|$ we obtain

$$|C_i| \leqslant \gamma_i L \int_0^t (a \cdot b) \exp\left\{\int_0^s a \cdot b\right\} ds \leqslant \gamma_i L \exp\left\{\int_0^t a \cdot b\right\}.$$

Similarly, setting

$$A_i(t) = \sum_j \left| \frac{d}{dt} \frac{a_j(t)}{a_i(t)} \right|$$

we obtain

$$\begin{aligned} |D_i| &\leqslant \int_0^t \exp\left\{\int_0^s a \cdot b\right\} a_i(s) \sum_j b_j(s) \int_0^t |z_i(\tau)|\, A_i(\tau)\, d\tau\, ds \\ &\leqslant \left(\int_0^t |z_i(s)|\, A_i(s)\, ds\right) \left(\int_0^t \exp\left\{\int_0^s a \cdot b\right\} L(a \cdot b)\, ds\right) \\ &\leqslant L \exp\left\{\int_0^t a \cdot b\right\} \int_0^t |z_i(s)|\, A_i(s)\, ds. \end{aligned}$$

Using these estimates in (9) there follows

$$|z_i(t)| \leqslant |z_i(0)| + \gamma_i L + L \int_0^t A_i(s)\, |z_i(s)|\, ds$$

to which we apply Gronwall's inequality to obtain finally

$$| z_i(t) | \leqslant (| z_i(0) | + \gamma_i L) \exp \left\{ L \int_0^t A_i(s)\, ds \right\}. \tag{10}$$

Now from (6) it follows that $\gamma_i \leqslant \text{const} \mid z_i(0) \mid$ and from (5) it follows that $\int_0^t A_i(s)\, ds$ is uniformly bounded for $t \geqslant 0$. Therefore, we have

$$| z_i(t) | \leqslant \text{const} \mid z_i(0) \mid .$$

From this we obtain (2) and the theorem is proved.

REMARK 1. A result similar to that of the above theorem can be obtained by replacing the assumption (5) concerning $a(t)$ with an analogous assumption concerning $b(t)$. In this case one first estimates the solution of the adjoint system

$$\frac{dy}{dt} = b(a \cdot y)$$

and then obtains the bound (2) from this estimate.

REMARK 2. The assumption that $a(t)$ is absolutely continuous is not necessary. All that is needed is that $a(t)$ be locally integrable and that each ratio, a_i/a_j, be of bounded variation on $0 \leqslant t < \infty$. In this case we can first approximate $a(t)$ by $a^h = a * \omega_h$ where ω_h is the Gaussian averaging kernel (mollifier) of radius h, [3]. The solutions $z^h(t)$ of the resulting "averaged" equations will satisfy (10) with $A_i = A_i^h$. Taking limits as $h \to 0$, the absolutely continuous solution $z(t)$ of (1) in the sense of Caratheodory [4] is then seen to satisfy (10) where now A_i is a measure i.e. $\int_0^t A_i\, ds$ is to be replaced by $\sum_j V_j(t)$ where $V_j(t)$ is the total variation of a_j/a_i over the interval $[0, t]$.

REMARK 3. The method we have used to prove this theorem can also be used to obtain a rather simple estimate for a nonnegative valued function $u(t)$ satisfying

$$u(t) \leqslant u_0(t) + \sum_{i=1}^{n} b_i(t) \int_0^t a_i(s)\, u(s)\, ds. \tag{11}$$

Here, $u_0(t)$ is a nonnegative locally integrable function and $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are as in the theorem. Using standard comparison theorems the problem is first reduced to that of equality in (11) in which case we have the system

$$\frac{dz_i}{dt} = a_i u_0 + a_i(b \cdot z) \qquad i = 1, \ldots, n$$

for the functions $z_i(t) = \int_0^t a_i(s)\, u(s)\, ds$. This inhomogeneous system with homogeneous initial data can be treated in a manner similar to that used in the proof of the above theorem for the homogeneous system (1) with inhomogeneous initial data. The resulting estimate for $z(t)$ then leads to the desired estimate for $u(t)$ [5].

2. An Example on Instability

We give an example of a system of the form (1) which is not stable on the right and for which (4), but not (5), holds. We take $n = 2$ and rewrite (1) in terms of plane polar coordinates,

$$r^{-1}\frac{dr}{dt} = -\mid a \mid \mid b \mid \cos(\alpha - \theta)\cos(\beta - \theta)$$

$$\frac{d\theta}{dt} = -\mid a \mid \mid b \mid \sin(\alpha - \theta)\cos(\beta - \theta), \tag{12}$$

where

$$z = (z_1, z_2) = (r\cos\theta, r\sin\theta), \qquad a = (a_1, a_2) = (\mid a \mid \cos\alpha, \mid a \mid \sin\alpha)$$

and

$$b = (b_1, b_2) = (\mid b \mid \cos\beta, \mid b \mid \sin\beta).$$

The quantity $r\sin(\alpha - \theta)$ is an integral of the motion (12) in any interval on which α is constant. In particular, then

$$r(t)\sin[\alpha - \theta(t)] = r(t_\nu)\sin[\alpha - \theta(t_\nu)], \qquad t \in [t_\nu, t_{\nu+1}), \tag{13}$$

for any interval $[t_\nu, t_{\nu+1})$ in which either

$$a = (1, 2\delta), \qquad b = (\delta, 1), \tag{14}$$

or

$$a = (2\delta, 1), \qquad b = (1, \delta). \tag{15}$$

Here δ is any fixed number which we take below to satisfy $0 < \delta < (\sqrt{2})^{-1}$.

Assume for the moment the existence of a sequence $\{t_\nu : t_\nu \in \mathbb{R}, \nu = 0, 1,\ldots\}$ and functions $r(t)$, $\theta(t)$ such that

$$t_0 = 0 < t_1 < \cdots < t_k < \cdots \qquad (\lim_{k\to\infty} t_k = \infty)$$

with r, θ continuous on $t \geqslant 0$ and satisfying (12) in each subinterval $(t_\nu, t_{\nu+1})$, where the vectors a and b in (12) are defined on the νth subinterval $[t_\nu, t_{\nu+1})$ by (14) or (15) respectively depending on whether ν is even or odd. Moreover assume

$$r(0) = r_0 > 0$$

and

$$\theta(t_\nu) = \begin{cases} \alpha_0 - \dfrac{\pi}{2} & \text{if} \quad \nu \text{ is even} \\ -\alpha_0 & \text{if} \quad \nu \text{ is odd,} \end{cases} \tag{16}$$

where α_0 is the fixed angle defined by $\tan \alpha_0 = 2\delta$, $0 < \alpha_0 < \pi/2$. We prove below the existence of such $\{t_\nu\}$, $r(t)$ and $\theta(t)$; assume this now and observe the consequences.

Since $\alpha \equiv \arctan a_2 a_1^{-1}$ has the constant value α_0 or $(\pi/2) - \alpha_0$ in $[t_\nu, t_{\nu+1})$ depending on whether ν is even or odd, there follows by (16) the result

$$\alpha(t_\nu) - \theta(t_\nu) = \frac{\pi}{2}$$

for all $\nu = 0, 1, \ldots$. Since $r(t)$ is continuous, (13) then yields

$$r(t_{\nu+1}) = \frac{r(t_\nu)}{\sin[\alpha(t_\nu) - \theta(t_{\nu+1})]}\,.$$

But $\alpha(t_\nu) - \theta(t_{\nu+1})$ equals $2\alpha_0$ for ν even and $\pi - 2\alpha_0$ for ν odd, so

$$r(t_{\nu+1}) = \Delta \cdot r(t_\nu), \qquad \text{all} \qquad \nu = 0, 1, \ldots,$$

with

$$\Delta = (\sin 2\alpha_0)^{-1} = [\sin(\pi - 2\alpha_0)]^{-1} = \frac{1 + 4\delta^2}{4\delta} > 1.$$

Hence $r = |z|$ satisfies $r(t_\nu) = \Delta^\nu r_0$ which is unbounded as $\nu \to \infty$ $(r_0 > 0)$. Since $\lim_{\nu\to\infty} t_\nu = \infty$, the system (1) so constructed is instable on the right.

Remarks. The example satisfies (4) with $K = \delta^{-1}$, but clearly neither of the vectors $a(t) \mid a(t) \mid^{-1}$ or $b(t) \mid b(t) \mid^{-1}$ is of bounded variation on $t \geqslant 0$. It is possible to mollify the transitions between (14) and (15) in the definition of $a(t)$ and $b(t)$ by considering their Gaussian averages so as to obtain a similar example with smooth data while maintaining instability. Similar examples can be constructed in any dimension $n \geqslant 2$.

Proof of the Existence of $\{t_\nu\}$, $r(t)$, $\theta(t)$. Suppose we have already t_0, $t_1, \ldots, t_K$ such that $t_0 = 0 < t_1 < \cdots < t_{K-1} < t_K$ with continuous functions $r(t)$ and $\theta(t)$ on $t_0 \leqslant t \leqslant t_K$ satisfying (12) except at the point t_ν $(\nu = 0, \ldots, K)$, and where the vectors $a(t)$ and $b(t)$ in (12) are defined on the νth subinterval $[t_\nu, t_{\nu+1})$ by (14) or (15) respectively depending on whether ν is even or odd. Then $r(t)$ and $\theta(t)$ can be continued continuously past $t = t_K$ as the unique solutions of (12) for $t > t_K$ agreeing with the given values of r

and θ at $t = t_K$, where again the vectors a and b in (12) are given for $t > t_K$ either by (14) or by (15) depending on whether K is even or odd. Such global continuations exist in view of the relation between (12) and the linear system (1). Suppose also that $r(t_0) = r_0 > 0$ and that (16) holds for $\nu = 0,\ldots, K$. It suffices under these conditions to show the existence of a number $t_{K+1} > t_K$ such that (16) holds also for $\nu = K + 1$, for we can then restrict the functions $r(t)$ and $\theta(t)$ to the interval $[t_0, t_{K+1}]$ and proceed as before, obtaining by induction the required sequence $\{t_\nu\}$ and functions $r(t)$, $\theta(t)$ defined on $t \geqslant 0$.

Assume first that K is even with the vectors a and b then given by (14) for $t \geqslant t_K$. Set $\phi(t) = \beta - \theta(t)$ and find from (12) the equation

$$\frac{d\phi}{dt} = | a | \, | b | \sin(\alpha - \beta + \phi) \cos \phi, \qquad t \geqslant t_K \tag{17}$$

where $\tan \alpha = 2\delta$, $\tan \beta = \delta^{-1}$ and $0 < \alpha$, $\beta < \pi/2$. One checks that

$$0 < \beta - \alpha < \frac{\pi}{2} < \beta + \alpha \tag{18}$$

which with (16) implies $\alpha - \beta + \phi(t_K) = \pi/2 < \phi(t_K) < \pi$. This result with (17) and another use of (18) implies $d\phi/dt \leqslant 0$ for all $t \geqslant t_K$ for which $\phi(t) \geqslant \pi/2$. Hence there holds $\pi/2 \leqslant \phi(t) < \pi$ and $\alpha \leqslant \alpha - \beta + \phi(t) \leqslant \pi/2$ for all such $t \geqslant t_K$, which with (17) then yields the following differential inequality,

$$\frac{d\phi}{dt} \leqslant | a | \, | b | \sin \alpha \cos \phi \tag{19}$$

for all $t \geqslant t_K$ for which $\phi(t) \geqslant \pi/2$. Standard comparison results with (19) imply

$$\phi(t) \leqslant \phi_0(t) \tag{20}$$

for all $t \geqslant t_K$ for which $\phi(t) \geqslant \pi/2$, where $\phi_0(t)$ denotes the unique solution of the following initial-value problem:

$$\frac{d\phi_0}{dt} = \lambda \cos \phi_0, \qquad t \geqslant t_K, \qquad \lambda = | a | \, | b | \sin \alpha = 2\delta \cdot \sqrt{1 + \delta^2}$$

$$\phi_0(t_K) = \phi(t_K) = \beta - \alpha + \frac{\pi}{2}. \tag{21}$$

One checks easily that ϕ_0 is determined by the relations

$$\cos \phi_0(t) = -2\{D \exp[\lambda(t - t_K)] + D^{-1} \exp[-\lambda(t - t_K)]\}^{-1}$$

$$\sin \phi_0(t) = \frac{-\cos \phi_0(t)}{2} \{D \exp[\lambda(t - t_K)] - D^{-1} \exp[-\lambda(t - t_K)]\}$$

$$D = \frac{1 + \cos(\beta - \alpha)}{\sin(\beta - \alpha)} = \frac{3\delta + \sqrt{(1 + \delta^2)(1 + 4\delta^2)}}{1 - 2\delta^2}. \tag{22}$$

We assume that $0 < \delta < (\sqrt{2})^{-1}$ so that $D > 0$. It then follows from (22) that $\phi_0(t) > \pi/2$ with $\lim_{t\to\infty} \phi_0(t) = \pi/2$, which with (20) implies that $\phi(t)$ must assume every value in the interval $(\pi/2, \phi(t_K)) = (\pi/2, \pi/2 + \beta - \alpha)$ for $t > t_K$. It is easily verified that the above choice of δ places $\beta + \alpha$ in this interval, proving then the existence of the required number $t_{K+1} > t_K$ such that $\phi(t_{K+1}) = \beta + \alpha$ (t_{K+1} is uniquely determined since $d\phi/dt \leqslant 0$ for all relevant $t \geqslant t_K$). Finally there holds $\theta(t_{K+1}) = \beta - \phi(t_{K+1}) = -\alpha = -\alpha_0$, so that (16) is satisfied for $\nu = K + 1$, K even.

In the case K odd we take a and b to be given by (15) for $t \geqslant t_K$ so that (17) holds with $\tan\alpha = (2\delta)^{-1}$, $\tan\beta = \delta$ and $0 < \alpha, \beta < \pi/2$. As before we now find $\alpha - \beta + \phi(t_K) = \pi/2 > \phi(t_K) > 0$, and $d\phi/dt \geqslant 0$ for all $t \geqslant t_K$ for which $\phi \leqslant \pi/2$. Hence for all such t there holds $0 < \phi(t) \leqslant \pi/2$ and $\alpha - \beta + \phi(t) < \pi - 2\beta$, so that the differential inequality (19) is replaced by

$$\frac{d\phi}{dt} \geqslant |a||b| \sin 2\beta \cos\phi \tag{23}$$

for all $t \geqslant t_K$ for which $\phi(t) \leqslant \pi/2$. Proceeding as before it is easy to verify that this inequality implies the existence of a (unique) number $t_{K+1} > t_K$ such that $\phi(t_{K+1}) = \beta + \alpha < \pi/2$, yielding $\theta(t_{K+1}) = -\alpha$. Since in this case $\alpha = (\pi/2) - \alpha_0$ it follows again that (16) holds for $\nu = K + 1$.

Finally, the length $t_{K+1} - t_K$ of the Kth subinterval $[t_K, t_{K+1})$ can be bounded below to show that $\lim_{K\to\infty} t_K = \infty$. For this purpose one uses the obvious differential inequalities

$$\frac{d\phi}{dt} \geqslant |a||b| \cos\phi \qquad \text{if} \qquad K \text{ even}$$

$$\frac{d\phi}{dt} \leqslant |a||b| \cos\phi \qquad \text{if} \qquad K \text{ odd}$$

which are related to (19) and (23) respectively. This completes the construction of the example.

Notes and References

1. Such systems occur for example as the variational system of a nonlinear system of the form

$$\frac{dx}{dt} = f(u(t, x))$$

where $u: \mathbb{R}^{n+1} \to \mathbb{R}^1$ and $f: \mathbb{R}^1 \to \mathbb{R}^n$. This in turn describes the characteristic field associated with the quasilinear partial differential equation,

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial_i} f_i(u) = 0.$$

A singular perturbation problem equivalent to the stability problem discussed in this paper for the linear system (1) arises in one approach to the problem of uniqueness of weak solutions of this partial differential equation c.f. E. CONWAY AND J. SMOLLER. *Arch. Rational Mech. Anal.* **23** (1967), 399–408.
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5. D. WILLETT, *Proc. Am. Math. Soc.* **16** (1965), 774–778, has obtained a very general estimate for nonnegative $u(t)$ satisfying (11). Although this estimate is valid even when (5) is not satisfied it must be stated inductively in terms of the composition of certain functional operators.